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# Transitivity of preferences: when does it matter?

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# Transitivity of preferences: when does it matter?

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**Abstract**

We define the empirical conditions on prices and incomes under which transitivity of preferences has specific testable implications. In particular, we set out necessary and sufficient requirements for budget sets under which consumption choices can violate SARP (Strong Axiom of Revealed Preferences) but not WARP (Weak Axiom of Revealed Preferences). As SARP extends WARP by additionally imposing transitive preferences, this effectively defines the conditions under which transitivity is separately testable. Our characterization has not only theoretical but also practical relevance, as transitivity conditions are known to substantially aggravate the computational burden of empirical revealed preference analysis. For finite datasets, our characterization takes the form of triangular conditions that must hold for all three-element subsets of normalized prices, and which are easy to verify in practice. For infinite datasets, we formally establish an intuitive connection between our characterization and the concept of Hicksian aggregation. We demonstrate the practical use of our conditions through two empirical illustrations.

**JEL Classification:** C14, DO1, D11, D12.

**Keywords:** revealed preferences, WARP, SARP, transitive preferences, testable implications, Hicksian aggregation

## 1 Introduction

For demand behavior under linear budget constraints, it is well established that transitivity of preferences has no empirical bite as long as there are no more than two goods. Rose (1958) provided a formal statement of this fact by showing that, in a two-goods setting, the Weak Axiom of Revealed Preference (WARP) is empirically equivalent to the Strong

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Axiom of Revealed Preferences (SARP).[1] As SARP extends WARP by (only) imposing the additional requirement that preferences must be transitive, this effectively implies that transitivity of preferences does not have specific testable implications.

This non-testability of transitivity has an intuitive analogue in terms of testable properties of Slutsky matrices, which are typically studied in differential analysis of continuous demand. Specifically, Slutsky symmetry is always satisfied by construction in situations with two goods and, thus, only negative semi-definiteness of the Slutsky matrix can be tested empirically in such instances. This directly complies with the two classic results of Samuelson (1938) and Houthakker (1950): Samuelson showed that demand is consistent with WARP only if compensated demand effects are negative, whereas Houthakker showed that a consumer behaves consistent with utility maximization (implying Slutsky symmetry in addition to Slutsky negativity) if and only if demand is consistent with SARP.[2] In a two-goods setting, the equivalence between WARP and SARP translates into non-testability of Slutsky symmetry (in contrast to negativity).

We can conclude that the (lack of) empirical content of transitivity with two goods is well understood by now. However, the question remains under which conditions transitivity is testable when there are more than two goods. In this respect, an intuitive starting point relates to the possibility of dimension-reduction that is based on Hicksian aggregation.[3] A set of goods can be represented by a Hicksian aggregate if the goods' relative prices remain fixed over decision situations. Thus, by verifying the empirical validity of constant relative prices, we can check whether the demand for multiple goods can be studied in terms of two Hicksian aggregates. If this happens to be the case, it immediately follows from Rose (1958)'s result that WARP and SARP will be empirically equivalent.

Clearly, the condition of constant relative prices will not be met in most real life settings, which provides the core motivation for our current study. Specifically, we establish the empirical conditions on prices and incomes that characterize the empirical bite of transitivity in a general situation with multiple goods. These conditions are necessary and sufficient for transitivity of preferences to have no specific testable implications. In other words, if (and only if) the conditions are met, then dropping transitivity will lead to exactly the same empirical conclusions. The fact that our conditions are defined in terms of budget sets, without requiring quantity information, is particularly convenient from a practical point of view. It makes it possible to check on the basis of given prices and incomes whether it suffices to (only) check WARP (instead of SARP) to verify consistency with utility maximization. Conversely, it characterizes the budget conditions under which transitivity of preferences has separate empirical implications and, thus, for which transitivity restrictions can potentially add value to the analysis.

Interestingly, we can show that our general characterization generates Rose (1958)'s

[1] Samuelson (1938) originally introduced the WARP as a basic consistency requirement on consumption behavior: if a consumer chooses a first bundle over a second one in a particular choice situation (characterized by a linear budget constraint), then (s)he cannot choose this second bundle over the first one in a different choice situation. Houthakker (1950) defined SARP as the extension of WARP with transitive preferences.

[2] See also Kihlstrom, Mas-Colell, and Sonnenschein (1976) for related discussion.

[3] See, for example, Varian (1992) for a general discussion on Hicksian aggregation. Lewbel (1996) presents related results on commodity aggregation under specific assumptions.

conclusion in the specific instance with two goods. Furthermore, we can establish an intuitive relation between our characterization and the Hicksian aggregation argument that we gave above. Specifically, when applying our characterization result to a continuous setting (with infinitely many price-income regimes), we obtain a condition that basically states that all prices must lie in a common two-dimensional plane. We show that this is formally equivalent to a setting where goods can be linearly aggregated into two composite commodities, which we can interpret as two Hicksian aggregates. As an implication, this also establishes that (in a continuous setting) Slutsky negativity entails symmetry if and only if prices satisfy this particular type of Hicksian aggregation.

The remainder of this paper unfolds as follows. Section 2 provides some further motivation for the theoretical and practical relevance of our findings by discussing their relation to the existing literature on WARP, SARP and transitive preferences. Section 3 first introduces some notation and basic definitions, and subsequently presents our main result as a generalization of Rose (1958)'s original result. Section 4 shows the connection between our characterization and Hicksian aggregation when the set of possible prices becomes infinite. Section 5 shows the practical use of our theoretical findings through two empirical illustrations. Finally, Section 6 concludes.

## 2 Theoretical and practical relevance

In the theoretical literature, the question whether, and under what conditions, WARP and SARP are empirically distinguishable has attracted considerable attention over the years. Shortly after Rose (1958)'s result on the equivalence between WARP and SARP for two goods, Gale (1960) constructed a counterexample showing that WARP and SARP may differ in settings with more than two goods. Since then, various authors have presented further clarifications and extensions of Gale's basic result (see, e.g., Shafer (1977); Peters and Wakker (1994); Heufer (2014)). In a similar vein, Uzawa (1960) showed that, if a demand function satisfies WARP together with some regularity condition, then it also satisfies SARP. However, Bossert (1993) put this result into perspective by demonstrating that, for continuous demand functions, Uzawa's regularity condition alone already implies SARP.

A main difference with our current contribution is that these previous studies typically exemplified the distinction between WARP and SARP by constructing hypothetical datasets (containing prices, incomes and consumption quantities) that satisfy WARP but violate SARP. Such datasets, however, might never be encountered in reality. In this sense, it leaves open the question whether the possibility to distinguish SARP from WARP is merely a theoretical curiosity or also an empirical regularity. Moreover, the datasets that are constructed do not define general conditions on budget sets (i.e. prices and incomes, without quantities) under which SARP and WARP are empirically equivalent (or, conversely, transitivity is separately testable).

Next, an important practical motivation for our theoretical analysis relates to the computational issues associated with the verification of revealed preference axioms. In particular, whether or not transitivity concerns are taken into account (i.e. SARP-based versus WARP-based) bears heavily on the computational burden of empirical revealed preference analysis. See, for example, the recent studies of Echenique, Lee, and Shum (2011), Kitamura and Stoye (2013) and Blundell et al. (2015) for specific instances illustrating the computational complexity of SARP-based analysis. In general, dropping transitivity can considerably alleviate the computational efforts needed for empirical applications. This consideration becomes all the more important given that increasingly large consumption datasets are becoming available. Attractively, our conditions are easy to verify in practice, even for such large datasets.

Finally, our results also have practical relevance from a noncomputational point of view. Since Tversky (1969)'s seminal paper on intransitivity of preferences, the realism of transitive preferences has become a popular research topic in both psychology and (behavioral) economics (see, e.g., Regenwetter, Dana, and Davis-Stober (2011) for an overview). Our findings can be useful for the design of experiments that aim at testing revealed preference axioms in a laboratory setting (in the tradition of Tversky (1969)). For example, one might be interested in separately testing transitivity of preferences. This requires budget sets for which SARP is not equivalent to WARP, which we characterize in our following analysis.

# 3 When WARP equals SARP

We assume a consumer who composes bundles of $m$ goods for $n$ budget sets. This defines a dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ with price (row) vectors $\mathbf{p}_t \in \mathbb{R}^m_{++}$ and quantity (column) vectors $\mathbf{q}_t \in \mathbb{R}^m_+$. To facilitate our further discussion, we summarize the budget conditions in terms of normalized prices, which implies total expenditures $\mathbf{p}_t \mathbf{q}_t = 1$ for all observations $t = 1, \ldots, n$. We can now define the basic revealed preference concept.

**Definition 1.** *The bundle $\mathbf{q}_t$ at observation $t$ is **revealed preferred** to the bundle $\mathbf{q}_v$ at observation $v$ if $\mathbf{p}_t \mathbf{q}_t (= 1) \geq \mathbf{p}_t \mathbf{q}_v$. We denote this as $\mathbf{q}_t R \mathbf{q}_v$.*

In words, $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$ if $\mathbf{q}_v$ was cheaper than $\mathbf{q}_t$ at the prices observed at $t$. Then, we have the following definitions of WARP and SARP.

**Definition 2.** *A dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates **WARP** if $R$ has a cycle of length 2, i.e. $\mathbf{q}_t R \mathbf{q}_v R \mathbf{q}_t$ and $\mathbf{q}_t \neq \mathbf{q}_v$.*

**Definition 3.** *A dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates **SARP** if $R$ has a cycle, i.e. $\mathbf{q}_t \; R \; \mathbf{q}_v R \; \mathbf{q}_s \; \ldots \; R \; \mathbf{q}_k \; R \; \mathbf{q}_t$ for some sequence of observations $t, v, s, \ldots, k$ and not all bundles $\mathbf{q}_t, \ldots, \mathbf{q}_k$ are identical.*

It is clear from the definitions that SARP consistency implies WARP consistency. We are interested in the reverse relationship: under which conditions does WARP imply SARP? Given our specific research question, we consider settings in which the empirical analyst does not necessarily observe the quantity choices, but only the normalized prices (i.e. budget sets). For the given normalized prices, we are interested in the possibility that there exist corresponding quantity bundles that imply a SARP or WARP violation. To this end, we use the following definition.

**Definition 4.** *A set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is said to be* ***WARP-reducible*** *if, for any set of quantities* $\{\mathbf{q}_t | t = 1, \ldots, n\}$ *for which* $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ *violates SARP, we also have that* $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ *violates WARP.*

To set the stage, we first repeat Rose (1958)'s original result, which says that WARP is always equivalent to SARP if the number of goods equals two (i.e. $m = 2$). We phrase this result in terms of the terminology that we introduced above.

**Proposition 1.** *If there are only two goods (i.e.* $m = 2$*), then any set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is WARP-reducible.*

Our main result will provide a generalization of Proposition 1. It makes use of the concept of a triangular configuration.

**Definition 5.** *A set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is a* ***triangular configuration*** *if, for any three price vectors* $\mathbf{p}_t, \mathbf{p}_v$ *and* $\mathbf{p}_k$ *(with* $t, v, k \in \{1, \ldots, n\}$*), there exists a number* $\lambda \in [0, 1]$ *and a permutation* $\sigma : \{t, v, k\} \to \{t, v, k\}$ *such that the following condition holds:*

$$\mathbf{p}_{\sigma(t)} \leq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)} \text{ or } \mathbf{p}_{\sigma(t)} \geq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)}.$$

Note that the inequalities in this definition are vector inequalities. As such, Definition 5 states that, for any three vectors, we need that there is a convex combination of two of the three prices that is either smaller or larger than the third price vector. Checking whether a set of prices is a triangular configuration merely requires verifying the linear inequalities in Definition 5 for any possible combination of three prices. Clearly, this is easy to do in practice, even if the number of observations (i.e. $n$) gets large.

We can show that the triangular conditions in Definition 5 are necessary and sufficient for WARP and SARP to be equivalent.[4]

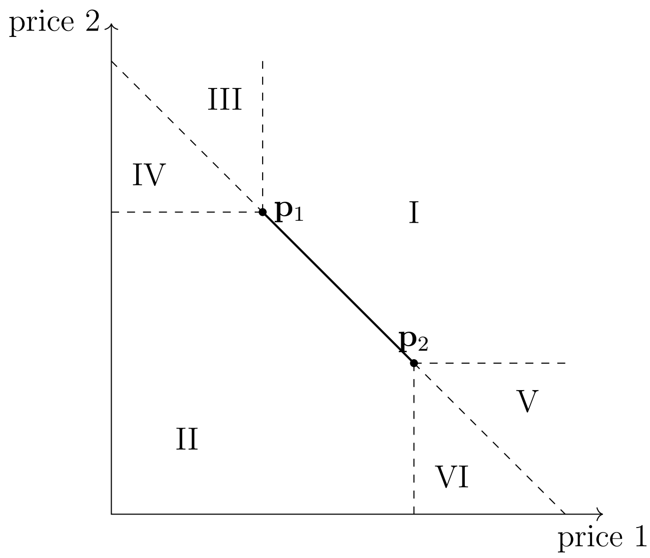
**Proposition 2.** *A set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is WARP-reducible if and only if it is a triangular configuration.*

This result generalizes Rose's result in Proposition 1. In particular, one can verify that, if the number of goods is equal to two, then any set of prices is a triangular configuration. To see this, consider three normalized price vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for two goods (i.e. $m = 2$). Obviously, if $\mathbf{p}_1 \geq \mathbf{p}_2$ or $\mathbf{p}_2 \geq \mathbf{p}_1$, we have that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a triangular configuration. Let us then consider the more interesting case where $\mathbf{p}_1$ and $\mathbf{p}_2$ are not ordered, which we illustrate in Figure 1. The price vector $\mathbf{p}_3$ should them fall into one of the six regions, which are numbered I to VI. For any of these six possible scenarios, the triangular condition in Definition 5 is met. To see this, we first consider the case where $\mathbf{p}_3$ lies in region I. In that case, $\mathbf{p}_3$ is obviously larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Similarly, if $\mathbf{p}_3$ lies in region II, it is smaller than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Next, if $\mathbf{p}_3$ lies in region III, then $\mathbf{p}_1$ is smaller than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ and, conversely, $\mathbf{p}_1$ is larger than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ if $\mathbf{p}_3$ lies in region IV. Finally, if $\mathbf{p}_3$ lies in region V, there is a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$ that dominates $\mathbf{p}_2$ and, if $\mathbf{p}_3$ lies in region

[4] The proofs of our main results are presented in Appendix A.

VI, then $\mathbf{p}_2$ is larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$. We can thus conclude that any possible set of prices $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is WARP-reducible.

Figure 1: The triangular condition in a two goods setting




Example 1 provides some further intuition for the result in Proposition 2. In this example, we focus on cycles of length 3, and show that the triangular configuration implies that each SARP violation of length 3 must contain a WARP violation.

**Example 1.** *Consider a set of three prices $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ that is a triangular configuration. Without loss of generality, we may assume that it is a triangular configuration because one of the following two inequalities holds: $\mathbf{p}_1 \leq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$ or $\mathbf{p}_1 \geq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$ for some $\lambda \in [0,1]$.*

*Let us first consider $\mathbf{p}_1 \leq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$. Assume that there exists a SARP violation with a cycle of length 3. With three observations, there are only two possibilities for cycles of length 3: $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$ or $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$. If $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$, then it must be that*

$$1 = \mathbf{p}_2\mathbf{q}_2 \geq \mathbf{p}_2\mathbf{q}_3 \text{ and } 1 = \mathbf{p}_3\mathbf{q}_3.$$

*Together with our triangular inequality this implies that*

$$1 \geq (\lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3)\mathbf{q}_3 \geq \mathbf{p}_1\mathbf{q}_3.$$

*As such, we can conclude that $\mathbf{q}_1 R \mathbf{q}_3$, which gives $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_1$, i.e. a violation of WARP. A similar reasoning holds for the second possibility (i.e. $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$), which shows that in this first case each violation of SARP implies a WARP violation.*

*For the second case, $\mathbf{p}_1 \geq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$, we must consider the same two possible SARP violations. The reasoning is now slightly different. In particular, let us assume that there is no violation of WARP. For the SARP violation $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$ this requires*

$1 < \mathbf{p}_3\mathbf{q}_2$ *(i.e. not* $\mathbf{q}_3 R \mathbf{q}_2$ *). Since* $1 = \mathbf{p}_2\mathbf{q}_2$*, we obtain that, if* $\lambda < 1$*,*

$$1 < (\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3)\mathbf{q}_2 \leq \mathbf{p}_1\mathbf{q}_2.$$

*This clearly contradicts* $\mathbf{q}_1 R \mathbf{q}_2$ *(i.e.* $1 \geq \mathbf{p}_1\mathbf{q}_2$*). If* $\lambda = 1$*, we have* $\mathbf{p}_1 \geq \mathbf{p}_2$ *and thus*

$$1 = \mathbf{p}_1\mathbf{q}_1 \geq \mathbf{p}_2\mathbf{q}_1.$$

*This again yields a contradiction, as it implies the WARP violation* $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_1$*. A similar reasoning holds for the second possibility (i.e.* $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$*), which shows that also for this case any SARP violation implies a WARP violation.*

# 4 Connection with Hicksian aggregation

So far we have assumed a finite data set with $n$ normalized prices (i.e. budget sets). This corresponds to a typical situation in empirical demand analysis, when the empirical analyst can only use a finite number of consumption observations. In this section, we consider the theoretical situation with a continuum of (normalized) prices. This will establish a formal connection between our triangular conditions and the notion of Hicksian aggregation. Specifically, we will show that, when the set of prices becomes infinite, our conditions converge to the requirement that the demand for multiple (i.e. $m$) goods can be summarized in terms of two Hicksian aggregates. In a sense, it establishes our characterization in Proposition 2 as a finite sample version of the Hicksian aggregation requirement for WARP to be equivalent to SARP.

To formalize the argument, let us consider an infinite set of prices $P$ such that, for all prices $\mathbf{p} \in P$ and all $\gamma > 0$, $\gamma\mathbf{p} \in P$. We remark that, because we focus on normalized prices (with total expenditures equal to unity), the price vector $\gamma\mathbf{p}$ equivalently corresponds to a situation with price vector $\mathbf{p}$ and total expenditures $1/\gamma$. In other words, our condition on the set $P$ actually allows us to consider any possible expenditure level for a given specification of prices.

Now consider the $n-1$ dimensional simplex $\Delta = \{\mathbf{p} \in \mathbb{R}^m_{++} | \sum_{i=1}^n (\mathbf{p})_i = 1\}$. Then, we can derive the next result.

**Proposition 3.** *Let* $P$ *be defined as above (i.e. if* $\mathbf{p} \in P$*, then* $\gamma\mathbf{p} \in P$ *for all* $\gamma > 0$*). If* $P \cap \Delta$ *is closed, then any three price vectors of* $P$ *satisfy the triangular condition if and only if there exist two vectors* $\mathbf{r}_1, \mathbf{r}_2 \in P$ *and, for all* $\mathbf{p} \in P$*, scalars* $\alpha, \beta \geq 0$ *that are not both zero, such that*

$$\mathbf{p} = \alpha \mathbf{r}_1 + \beta \mathbf{r}_2,$$

Basically, this result requires that all prices $\mathbf{p} \in P$ must lie in a common two-dimensional plane. The additional requirement that $P \cap \Delta$ is closed is a technical condition guaranteeing that $\mathbf{r}_1$ and $\mathbf{r}_2$ belong to $P$.

Interestingly, Proposition 3 allows us to interpret our triangular conditions (under infinitely many prices) in terms of Hicksian quantity aggregation. Specifically, Hicksian aggregation requires that all prices in a subset of goods change proportionally to some common price vector (i.e. $\mathbf{p} = \alpha\mathbf{r}$ for all $t$, with $\mathbf{r} \in \mathbb{R}^m_+$ and scalar $\alpha > 0$). In our case, we can, for any bundle $\mathbf{q}_t$, construct two new "aggregate quantities" $z_{t,1} = \mathbf{r}_1\mathbf{q}_t$ and $z_{t,2} = \mathbf{r}_2\mathbf{q}_t$, to define the quantity bundle $\mathbf{z}_t = [z_{t,1}, z_{t,2}]$. Correspondingly, we can construct new "price vectors" $\mathbf{w}_t = [\alpha_t, \beta_t]$. Then, for any two observations $t$ and $v$, we have

$$1 \geq \mathbf{p}_t\mathbf{q}_v = (\alpha_t\mathbf{r}_1 + \beta_t\mathbf{r}_2)\mathbf{q}_v = \alpha_t\mathbf{r}_1\mathbf{q}_v + \beta_t\mathbf{r}_2\mathbf{q}_v = \mathbf{w}_t\mathbf{z}_v.$$

In other words, we obtain $\mathbf{q}_t R \mathbf{q}_v$ for the dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ if and only if $\mathbf{z}_t R \mathbf{z}_v$ for the dataset $\{(\mathbf{w}_t, \mathbf{z}_t)|t = 1, \ldots, n\}$. This implies that the dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ will violate SARP (resp. WARP) if and only if the dataset $\{(\mathbf{w}_t, \mathbf{z}_t)|t = 1, \ldots, n\}$ violates SARP (resp. WARP). Moreover, the dataset $\{(\mathbf{w}_t, \mathbf{z}_t)|t = 1, \ldots, n\}$ only contains two goods, so Proposition 1 implies that WARP is equivalent to SARP, and this equivalence carries over to the dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$. Basically, this defines the possibility to construct two Hicksian aggregates as a necessary and sufficient condition for WARP to be equivalent to SARP when there are infinitely many prices.

By building further on this intuition, we can also directly interpret Proposition 3 in terms of utility maximizing behavior. To see this, we start by considering a rational (i.e. SARP-consistent) individual with indirect utility function $v(\mathbf{p})$, which defines the maximal attainable utility for the (normalized) prices $\mathbf{p}$. By construction, this function $v(\mathbf{p})$ is quasi-convex, decreasing and satisfies Roy's identity, i.e. the $m$-dimensional demand functions are given by $\mathbf{q} = \frac{\nabla_\mathbf{p} v(\mathbf{p})}{\mathbf{p}\nabla_\mathbf{p} v(\mathbf{p})}$. By using our above notation, if the Hicksian aggregation property in Proposition 3 is satisfied, we can write $v(\mathbf{p}) = v(\alpha\mathbf{r}_1 + \beta\mathbf{r}_2) \equiv \tilde{v}(\alpha, \beta) = \tilde{v}(\mathbf{w})$. It is easy to verify that also $\tilde{v}(\mathbf{w})$ is quasi-convex, decreasing and satisfies Roy's identity, which in this case states that the two-dimensional demand functions satisfy $\mathbf{z} = \frac{\nabla_\mathbf{w} \tilde{v}(\mathbf{w})}{\mathbf{w}\nabla_\mathbf{w} \tilde{v}(\mathbf{w})}$.

# 5 Empirical illustrations

To show the practical relevance of our triangular conditions, we present empirical applications that make use of two different types of household datasets that have been the subject of empirical revealed preference analysis in recent studies. They will illustrate alternative possible uses of our characterization in Proposition 2.

**Panel data.** Our first application considers household data that are drawn from the Spanish survey ECPF (Encuesta Continua de Presupestos Familiares), which has been used in various SARP-based empirical analyses.[5] In what follows, we will specifically focus on the dataset that was studied by Beatty and Crawford (2011). This dataset contains a time-series of 8 observations for 1585 households, on 15 nondurable goods. Importantly, different households can be characterized by other price regimes, which makes that the empirical content of our triangular conditions will vary over households.

[5]See, for example, Crawford (2010), Beatty and Crawford (2011), Demuynck and Verriest (2013), Adams, Cherchye, De Rock, and Verriest (2014) and Cherchye, Demuynck, De Rock, and Hjertstrand (2014).

We begin by verifying whether the household-specific price series satisfy the conditions for two-dimensional Hicksian aggregation as we defined them in Section 4 (Proposition 3). As discussed before, these conditions are sufficient (but not necessary) for WARP to be equivalent to SARP in the case of finite datasets. It turns out that none of the 1585 household datasets satisfies the conditions. This shows that the Hicksian aggregation criteria are very stringent from an empirical point of view. More generally, it suggests that, for finite datasets, there is little hope that Hicksian aggregation arguments will provide an effective basis to justify a WARP-based empirical analysis instead of a SARP-based analysis.

By contrast, if we check the triangular conditions in Definition 5, we conclude that no less than 69.34% of the datasets satisfies these requirements. For these datasets, a WARP-based analysis is equally informative as a SARP-based analysis. In view of the computational burden associated with the transitivity requirement that is captured by SARP, we see this as quite a comforting conclusion from a practical point of view. It also indicates that the (necessary and sufficient) triangular conditions provide a substantially more useful basis than the (sufficient) Hicksian aggregation conditions to empirically support a WARP-based analysis. Even though the two types of conditions converge for infinitely large datasets, their empirical implications for finite datasets can differ considerably.

**Repeated cross-sectional data.** Our second application uses the data from the British Family Expenditure Survey (FES) that have been analysed by Blundell et al. (2003, 2008, 2015). These authors developed methods to combine Engel curves with revealed preference axioms to obtain tight bounds on cost of living indices and demand responses. These methods become substantially more elaborate when considering SARP instead of WARP. This makes it directly relevant to check whether WARP and SARP are equivalent for the budget sets taken up in the analysis.[6]

More specifically, the dataset is a repeated cross-section that contains 25 yearly observations (1975 to 1999) for three product categories (food, other nondurables and services). As in the original studies, we focus on mean income for each observation year. When checking our triangular conditions for all triples of (normalized) prices, we conclude that 2.39% of these triples violate these conditions. This indicates that WARP and SARP are not fully equivalent for these data. However, for a fraction as low as 2.39%, it is also fair to conclude that the subset of prices that may induce differences between WARP and SARP is quite small.

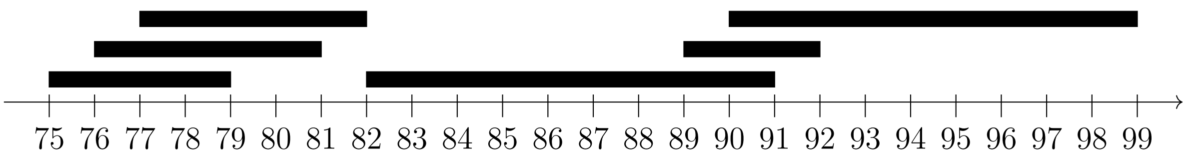
As a further exercise, we identified the largest subset of the 25 observation years that does satisfy the triangular conditions in Proposition 2.[7] It turns out that this largest

[6] In this respect, Kitamura and Stoye (2013) use the same FES data in their application of so-called "stochastic" axioms of revealed preference, which form the population analogues of the more standard revealed preference axioms such as WARP and SARP (see McFadden (2005) for an overview). In a stochastic revealed preference setting, the verification of WARP is relatively easy from a computational point of view (see, for example, Hoderlein and Stoye (2014) and Cosaert and Demuynck (2014)), while the verification of SARP is known to be difficult (i.e. NP-hard). As a direct implication, the knowledge that WARP is empirically equivalent to SARP can have a huge impact on the computation time.

[7] This subset can be identified by solving a simple integer programming problem (with binary integer variables). The program is available upon request.

triangular-consistent subset contains 17 observed budget sets. Putting it differently, if we drop 8 of the 25 original observations, we know that WARP-based and SARP-based analyses will obtain exactly the same conclusions.

Figure 2: Largest triangular consistent subperiods (FES)




As a last exercise, we redid the previous analysis but now focusing on continuous subperiods of the full period 1985-1999 that are consistent with our triangular conditions. This can provide guidance, for example, for breaking up the total set of observations into subsets, to subsequently conduct a separate WARP-based (or, equivalently, SARP-based) analysis for every other subset. The results of this exercise are reported in Figure 2. It turns out that the longest subperiods for which WARP and SARP are equivalent contain ten years (1982-1991 and 1990-1999). By contrast, the shortest continuous subperiod that satisfies our triangular conditions has only four years (1989-1992).

# 6 Conclusion

We have presented triangular conditions for budget sets that are necessary and sufficient for WARP and SARP to be empirically equivalent. This defines the empirical conditions under which transitivity of preferences has separate testable implications. Conveniently, our triangular conditions are easy to check in practice. From an empirical point of view, our conditions can be particularly relevant in settings where a SARP-based analysis requires substantially more computational effort than a WARP-based analysis. We clarified the formal connection between our characterization and the concept of Hicksian aggregation. We also conducted two empirical applications that illustrate alternative possible uses of our conditions.

# A Proofs

## A.1 Proof of Proposition 2

Before we give the proof of our main result, let us introduce some notation. For a finite set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$, consider the **convex hull** of $P$,

$$C(P) = \left\{ \mathbf{p} \in \mathbb{R}^m_{++} \middle| \mathbf{p} = \sum_{t=1}^{n} \alpha_t \mathbf{p}_t, \alpha_t \geq 0, \sum_{t=1}^{n} \alpha_t = 1 \right\},$$

and the **convex monotone hull** of $P$,

$$CM(P) = \left\{ \mathbf{p} \in \mathbb{R}^m_{++} \middle| \mathbf{p} \geq \sum_{t=1}^{n} \alpha_t \mathbf{p}_t, \alpha_t \geq 0, \sum_{t=1}^{n} \alpha_t = 1 \right\}.$$

The set $C(P)$ contains all prices that are a convex combination of the prices in $P$, while the set $CM(P)$ contains all prices that are at least as large as a convex combination of the prices in $P$. A price vector $\mathbf{p}_t$ is a called a **vertex** of $CM(P)$ if $\mathbf{p}_t \notin CM(P \setminus \{\mathbf{p}_t\})$. It is easy to verify that every element in $CM(P)$ is larger than or equal to some convex combination of the vertices of $CM(P)$.

Consider a bundle $\mathbf{q} \in \mathbb{R}^m_+$ that satisfies $\mathbf{p}_t \mathbf{q} = 1$. Then, the set of vectors $\mathbf{p}$

$$H(\mathbf{q}) = \{\mathbf{p} | \mathbf{p}\mathbf{q} = 1\},$$

defines an $(m-1)-$dimensional hyperplane in the space $\mathbb{R}^m$. Of course, we have that $\mathbf{p}_t \in H(\mathbf{q})$. For a non-zero vector $\mathbf{q} \in \mathbb{R}^m_+$, the hyperplane $H(\mathbf{q})$ is said **to cut** the set $C$ if there are two vectors $\mathbf{p}, \mathbf{p}' \in C$ such that $1 \leq \mathbf{p}\mathbf{q}$ and $1 \geq \mathbf{p}'\mathbf{q}$. If $C$ is non-empty and monotone (i.e. if $\mathbf{p} \in C$ and $\mathbf{p}' \geq \mathbf{p}$, then $\mathbf{p}' \in C$), then we can always find a vector $\mathbf{p}$ that satisfies the first inequality. In this case, only the second inequality is relevant.

Finally, for a number $j$ we write $\lfloor j \rfloor$ for $(j \mod n)$. We start by proving two lemmata.

**Lemma 1.** *Consider a set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$ and a non-zero consumption bundle $\mathbf{q}$ where $\mathbf{p}_t \mathbf{q} = 1$. If the hyperplane $H(\mathbf{q})$ cuts $CM(P) \setminus \{\mathbf{p}_t\}$, then there is a vertex $\mathbf{p}_v \in CM(P)$, distinct from $\mathbf{p}_t$, such that $1 \geq \mathbf{p}_v \mathbf{q}$.*

*Proof.* If $H(\mathbf{q})$ cuts $CM(P) \setminus \{\mathbf{p}_t\}$, then there is a vector $\mathbf{p} \in CM(P)$, with $\mathbf{p} \neq \mathbf{p}_t$, such that $1 \geq \mathbf{p}\mathbf{q}$. From the definition of $CM(P)$ there must exist numbers $\alpha_j \geq 0$, with $\sum_j \alpha_j = 1$ and

$$1 \geq \mathbf{p}\mathbf{q} \geq \left( \sum_{j=1}^{n} \alpha_j \mathbf{p}_j \right) \mathbf{q} = \sum_{j=1}^{n} \alpha_j \mathbf{p}_j \mathbf{q},$$

As mentioned above, without loss of generality, we may assume that all $\mathbf{p}_j$ corresponding to a strict positive $\alpha_j$ are vertices.

Let $J = \arg\min_j \mathbf{p}_j \mathbf{q}$ where $j$ is restricted to those values with $\alpha_j > 0$. If there is a $j \in J$ with $\mathbf{p}_j \neq \mathbf{p}_t$, we obtain that $1 \geq \mathbf{p}_j \mathbf{q}$ what we needed to proof. On the other hand, if $J = \{t\}$, then (from $\mathbf{p}_t \neq \mathbf{p}$)

$$1 \geq \mathbf{p}\mathbf{q} \geq \sum_{j=1}^{n} \alpha_j \mathbf{p}_j \mathbf{q} > \mathbf{p}_t \mathbf{q}.$$

This gives a contradiction with $\mathbf{p}_t \mathbf{q} = 1$. □

The following lemma is similar to Theorem 1 in Heufer (2014), but it is stated in terms of prices instead of quantities.

**Lemma 2.** *Let $P = \{\mathbf{p}_t | t = 1, \dots, n\}$ be a set of prices and let $\{\mathbf{q}_t | t = 1, \dots, n\}$ be a set of distinct non-zero bundles such that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \dots, n\}$ violates SARP. Also assume that no strict subset of $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \dots, n\}$ violates SARP. Without loss of generality, assume that the SARP violation is given by $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 \dots R \mathbf{q}_n R \mathbf{q}_1$ (i.e. $1 \geq \mathbf{p}_1 \mathbf{q}_2, 1 \geq \mathbf{p}_2 \mathbf{q}_3, \dots, 1 \geq \mathbf{p}_n \mathbf{q}_1$). Then,*

1. *the prices in $P$ are the vertices of the set $CM(P)$;*

2. *for all $\alpha \in ]0, 1[$ and $j = 1, \dots, n$ the vector $\alpha \mathbf{p}_j + (1 - \alpha) \mathbf{p}_{\lfloor j+1 \rfloor}$ is not in the relative interior of $CM(P)$, i.e. there do not exist numbers $\alpha_t \geq 0, \sum_t \alpha_t = 1$ such that,*

$$\alpha \mathbf{p}_j + (1 - \alpha) \mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t=1}^{n} \alpha_t \mathbf{p}_t.$$

*Proof.* Assume, towards a contradiction, that $\mathbf{p}_j$, with $j \in \{1, \dots, n\}$, is not a vertex of $CM(P)$. Since $CM(P)$ is monotone and $1 \geq \mathbf{p}_j \mathbf{q}_{\lfloor j+1 \rfloor}$, we know that the hyperplane $H(\mathbf{q}_{\lfloor j+1 \rfloor})$ cuts $CM(P)$. From Lemma 1 we therefore obtain that there exists some vertex $\mathbf{p}_v$ of $CM(P)$, such that $1 \geq \mathbf{p}_v \mathbf{q}_{\lfloor j+1 \rfloor}$. As $\mathbf{p}_j$ is not a vertex, $\mathbf{p}_v \neq \mathbf{p}_j$. If $v < j < n$, we have that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \dots, v, j + 1, \dots, n\}$ violates SARP. If $v < j = n$, we have that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \dots, v\}$ violates SARP. Finally, if $v > j$, we obtain that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = j + 1, \dots, v\}$ violates SARP. In all cases we thus obtain the desired contradiction as a strict subset of $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \dots, n\}$ violates SARP.

For the second part, assume, again towards a contradiction, that there exists an $\alpha \in ]0, 1[$ such that $\mathbf{p}' = \alpha \mathbf{p}_j + (1 - \alpha) \mathbf{p}_{\lfloor j+1 \rfloor}$ is in the relative interior of $CM(P)$. That is, there exist numbers $\alpha_t \geq 0$, with $\sum_{t=1}^{n} \alpha_t = 1$, such that

$$\alpha \mathbf{p}_j + (1 - \alpha) \mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t=1}^{n} \alpha_t \mathbf{p}_t.$$

Observe that $\alpha_t > 0$ for at least one $t \notin \{j, \lfloor j + 1 \rfloor\}$, since $\mathbf{p}_j$ and $\mathbf{p}_{\lfloor j+1 \rfloor}$ are both vertices of CM(P). Rewriting this inequality gives

$$(\alpha - \alpha_j) \mathbf{p}_j + (1 - \alpha - \alpha_{\lfloor j+1 \rfloor}) \mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \alpha_t \mathbf{p}_t,$$

where $\alpha_j$ or $\alpha_{\lfloor j+1 \rfloor}$ are potentially equal to zero. Given that the right hand side is strictly positive, one of the terms $(\alpha - \alpha_j)$ or $(1 - \alpha - \alpha_{\lfloor j+1 \rfloor})$ should be strictly positive. If the first term is strictly positive and the second term is non-positive, then

$$\mathbf{p}_j > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \frac{\alpha_t}{\alpha - \alpha_j} \mathbf{p}_t + \frac{\alpha - 1 + \alpha_{\lfloor j+1 \rfloor}}{\alpha - \alpha_j} \mathbf{p}_{j+1}.$$

This shows that $\mathbf{p}_j$ is in $CM(P \setminus \{\mathbf{p}_j\})$, a contradiction with the first part of the lemma. Similarly, if the first term is non-positive and the second term strictly positive, then

$$\mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \frac{\alpha_t}{1 - \alpha - \alpha_{\lfloor j+1 \rfloor}} \mathbf{p}_t + \frac{\alpha_j - \alpha}{1 - \alpha - \alpha_{\lfloor j+1 \rfloor}} \mathbf{p}_j.$$

Now we have that $\mathbf{p}_{\lfloor j+1 \rfloor} \in CM(P \setminus \{\mathbf{p}_{\lfloor j+1 \rfloor}\})$, again a contradiction with the first part of the lemma. Finally, if both terms are strictly positive, then

$$\frac{(\alpha - \alpha_j)\mathbf{p}_j + (1 - \alpha - \alpha_{\lfloor j+1 \rfloor})\mathbf{p}_{\lfloor j+1 \rfloor}}{1 - \alpha_j - \alpha_{\lfloor j+1 \rfloor}} > \sum_{t=1, t \notin \{j, \lfloor j+1 \rfloor\}}^{n} \frac{\alpha_j}{1 - \alpha_j - \alpha_{\lfloor j+1 \rfloor}} \mathbf{p}_t.$$

Denote the left hand side by $\mathbf{p}'''$, then the above inequality shows that $\mathbf{p}''' \in CM(P) \setminus \{\mathbf{p}_j\}$. Moreover, as $1 \geq \mathbf{p}_j \mathbf{q}_{\lfloor j+1 \rfloor}$ and $1 = \mathbf{p}_{\lfloor j+1 \rfloor} \mathbf{q}_{\lfloor j+1 \rfloor}$, we have that $1 \geq \mathbf{p}''' \mathbf{q}_{\lfloor j+1 \rfloor}$, as $\mathbf{p}'''$ is a weighted average of both $\mathbf{p}_j$ and $\mathbf{p}_{\lfloor j+1 \rfloor}$. This shows that $H(\mathbf{q}_{\lfloor j+1 \rfloor})$ cuts the set $CM(P) \setminus \{\mathbf{p}_j\}$. Similar to before, we can thus use Lemma 1 (i.e. there exist a vertex $\mathbf{p}_v \in CM(P)$ distinct from $\mathbf{p}_j$ such that $1 \geq \mathbf{p}_v \mathbf{q}_{\lfloor j+1 \rfloor}$) to conclude that there must exists a strictly smaller subset of prices that implies a violation of SARP, which again gives us the desired contradiction. □

We can then prove Proposition 2.

*Proof.* **Sufficiency.** Consider a set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$ that satisfy the triangular configuration condition. If for all sets of bundles $\mathbf{q}_1, \ldots, \mathbf{q}_n$, $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ satisfies SARP, then evidently, WARP is also satisfied, so there is nothing left to prove. Therefore consider a set $\{\mathbf{q}_t | t = 1, \ldots, n\}$ of distinct bundles such that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates SARP and assume, towards a contradiction, that it satisfies WARP. Note that we may consider the case where $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ contains no smaller subset that also violate SARP (since otherwise we could replace $P$ by a smaller subset of prices). Next, let us renumber the observations such that the SARP violation is given by $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 \ldots R \mathbf{q}_n R \mathbf{q}_1$ (i.e. $1 \geq \mathbf{p}_1 \mathbf{q}_2, 1 \geq \mathbf{p}_2 \mathbf{q}_3, \ldots, 1 \geq \mathbf{p}_n \mathbf{q}_1$).

Consider all three element subsets $\{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}\}$. Given that $P$ is a triangular configuration, we have that, for all $j$, there is a $\lambda \in [0, 1]$ such that one of the following inequalities holds:

$$\mathbf{p}_j \leq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1 - \lambda) \mathbf{p}_{\lfloor j+2 \rfloor}, \quad (1)$$

$$\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda \mathbf{p}_j + (1 - \lambda) \mathbf{p}_{\lfloor j+2 \rfloor}, \quad (2)$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \leq \lambda \mathbf{p}_j + (1 - \lambda) \mathbf{p}_{\lfloor j+1 \rfloor}, \quad (3)$$

$$\mathbf{p}_j \geq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1 - \lambda) \mathbf{p}_{\lfloor j+2 \rfloor}, \quad (4)$$

$$\mathbf{p}_{\lfloor j+1 \rfloor} \geq \lambda \mathbf{p}_j + (1 - \lambda) \mathbf{p}_{\lfloor j+2 \rfloor}, \quad (5)$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \geq \lambda \mathbf{p}_j + (1 - \lambda) \mathbf{p}_{\lfloor j+1 \rfloor}. \quad (6)$$

If one of the latter inequalities (4)-(6) holds, then either $\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}$ or $\mathbf{p}_{\lfloor j+2 \rfloor}$ is not a vertex of $CM(P)$, which contradicts Lemma 2. Given this, it must be that one of the inequalities (1)-(3) holds. Let us first show that (1) and (3) cannot hold.

Assume that (1) holds. Since $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ contains no subset that violates SARP, we know from Lemma 2 that the inequality cannot be strict. As such, it must be that the inequality holds with equality, i.e.

$$\mathbf{p}_j = \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1 - \lambda) \mathbf{p}_{\lfloor j+2 \rfloor}.$$

This implies that

$$\begin{aligned}\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} &= \lambda\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+2\rfloor}\\ &\leq \lambda\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda)\\ &\leq 1.\end{aligned}$$

As such we obtain that the smaller dataset $\{(\mathbf{p}_t,\mathbf{q}_t)|t=1,\dots n; t\neq\lfloor j+1\rfloor\}$ violates SARP, a contradiction.

Next assume that (3) holds. Again by Lemma 2, we have that the inequality cannot be strict and thus that

$$\mathbf{p}_{\lfloor j+2\rfloor} = \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}.$$

Observe that $1 < \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}$, since otherwise $\{\mathbf{p}_{\lfloor j+1\rfloor},\mathbf{q}_{\lfloor j+1\rfloor},\mathbf{p}_{\lfloor j+2\rfloor},\mathbf{q}_{\lfloor j+2\rfloor}\}$ violates WARP. This implies

$$\begin{aligned}1 &< \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda),\end{aligned}$$

which is equivalent to $1 < \mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor}$. This contradicts $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor}$.

We can thus conclude that for all $j$, (2) must hold. If for some $j$ this inequality holds with an equality, then

$$\begin{aligned}1 &\geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+2\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+2\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda),\end{aligned}$$

which implies that $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor}$. But then $\{(\mathbf{p}_t,\mathbf{q}_t)|t=1,\dots,n; t\neq\lfloor j+1\rfloor\}$ violates SARP, which gives a contradiction.

Given all this, it must be the case that there exist numbers $\lambda_j\in[0,1]$ such that,

$$\begin{aligned}&\lambda_1\mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3 > \mathbf{p}_2,\\ &\lambda_2\mathbf{p}_2 + (1-\lambda_2)\mathbf{p}_4 > \mathbf{p}_3,\\ &\dots,\\ &\lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_{n-1})\mathbf{p}_1 > \mathbf{p}_n,\\ &\lambda_n\mathbf{p}_n + (1-\lambda_n)\mathbf{p}_2 > \mathbf{p}_1.\end{aligned}$$

Let us first show that for all $j$, $\lambda_j\notin\{0,1\}$. If $\lambda_j=1$, we obtain that $\mathbf{p}_j > \mathbf{p}_{\lfloor j+1\rfloor}$. Then $1=\mathbf{p}_j\mathbf{q}_j \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_j$ so we have that $\{\mathbf{p}_j,\mathbf{q}_j,\mathbf{p}_{\lfloor j+1\rfloor},\mathbf{q}_{\lfloor j+1\rfloor}\}$ violates WARP. If $\lambda_j=0$, we obtain that $\mathbf{p}_{\lfloor j+2\rfloor} > \mathbf{p}_{\lfloor j+1\rfloor}$. Then we have that $1 \geq \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+3\rfloor} \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+3\rfloor}$. This implies that the smaller dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n; t \neq \lfloor j+2 \rfloor\}$ violates SARP.

Now, let us show by induction on $n$ that above system of inequalities with $\lambda_j \in ]0,1[$ can not have a solution for the $\lambda_j$.

If $n = 3$, we obtain the system,

$$\begin{aligned} \lambda_1 \mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3 &> \mathbf{p}_2, \\ \lambda_2 \mathbf{p}_2 + (1-\lambda_2)\mathbf{p}_1 &> \mathbf{p}_3, \\ \lambda_3 \mathbf{p}_3 + (1-\lambda_3)\mathbf{p}_2 &> \mathbf{p}_1. \end{aligned}$$

This gives

$$\begin{aligned} \mathbf{p}_2 &< \lambda_1 \mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3, \\ \mathbf{p}_2 &> -\frac{(1-\lambda_2)}{\lambda_2}\mathbf{p}_1 + \frac{1}{\lambda_2}\mathbf{p}_3, \\ \mathbf{p}_2 &> -\frac{\lambda_3}{1-\lambda_3}\mathbf{p}_3 + \frac{1}{1-\lambda_3}\mathbf{p}_1. \end{aligned}$$

Combining these inequalities leads to

$$\begin{aligned} 0 &< \left(\lambda_1 + \frac{(1-\lambda_2)}{\lambda_2}\right)\mathbf{p}_1 + \left((1-\lambda_1) - \frac{1}{\lambda_2}\right)\mathbf{p}_3, \\ 0 &< \left(\lambda_1 - \frac{1}{1-\lambda_3}\right)\mathbf{p}_1 + \left((1-\lambda_1) + \frac{\lambda_3}{1-\lambda_3}\right)\mathbf{p}_3, \end{aligned}$$

which gives the contradiction,

$$\mathbf{p}_1 > \mathbf{p}_3 \text{ and } \mathbf{p}_1 < \mathbf{p}_3.$$

For the induction step, assume that there is no solution for any set of $n$ prices and consider a system of inequalities with $n+1$ prices. The inequalities involving $\mathbf{p}_{n+1}$ are given by

$$\begin{aligned} \lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_{n-1})\mathbf{p}_{n+1} &> \mathbf{p}_n, \\ \lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 &> \mathbf{p}_{n+1}, \\ \lambda_{n+1}\mathbf{p}_{n+1} + (1-\lambda_{n+1})\mathbf{p}_2 &> \mathbf{p}_1. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \mathbf{p}_{n+1} &> \frac{1}{1-\lambda_{n-1}}\mathbf{p}_n - \frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1}, \\ \mathbf{p}_{n+1} &< \lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1, \\ \mathbf{p}_{n+1} &> -\frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 + \frac{1}{\lambda_{n+1}}\mathbf{p}_1 \end{aligned}$$

Combining these inequalities leads to

$$\begin{aligned}
&\lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 > \frac{1}{1-\lambda_{n-1}}\mathbf{p}_n - \frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1},\\
&\lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 > -\frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 + \frac{1}{\lambda_{n+1}}\mathbf{p}_1\\
\Longleftrightarrow\ &\frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1} + (1-\lambda_n)\mathbf{p}_1 > \left(\frac{1}{1-\lambda_{n-1}} - \lambda_n\right)\mathbf{p}_n,\\
&\lambda_n \mathbf{p}_n + \frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 > \left(\frac{1}{\lambda_{n+1}} - (1-\lambda_n)\right)\mathbf{p}_1\\
\Longleftrightarrow\ &\lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_n)(1-\lambda_{n-1})\mathbf{p}_1 > (1-\lambda_n(1-\lambda_{n-1}))\,\mathbf{p}_n,\\
&\lambda_n\lambda_{n+1}\mathbf{p}_n + (1-\lambda_{n+1})\mathbf{p}_2 > (1-(1-\lambda_n)\lambda_{n+1})\,\mathbf{p}_1\\
\Longleftrightarrow\ &\frac{\lambda_{n-1}}{1-\lambda_n(1-\lambda_{n-1})}\mathbf{p}_{n-1} + \frac{(1-\lambda_n)(1-\lambda_{n-1})}{1-\lambda_n(1-\lambda_{n-1})}\mathbf{p}_1 > \mathbf{p}_n,\\
&\frac{\lambda_n\lambda_{n+1}}{1-(1-\lambda_n)\lambda_{n+1}}\mathbf{p}_n + \frac{(1-\lambda_{n+1})}{1-(1-\lambda_n)\lambda_{n+1}}\mathbf{p}_2 > \mathbf{p}_1.
\end{aligned}$$

Let us denote $\lambda'_{n-1} = \frac{\lambda_{n-1}}{1-\lambda_n(1-\lambda_{n-1})}$ and $\lambda'_n = \frac{\lambda_n\lambda_{n+1}}{1-(1-\lambda_n)\lambda_{n+1}}$. It is easily verified that $\lambda'_{n-1}, \lambda'_n \in ]0,1[$. Substitution then gives

$$\lambda'_{n-1}\mathbf{p}_{n-1} + (1-\lambda'_{n-1})\mathbf{p}_1 > \mathbf{p}_n \text{ and } \lambda'_n\mathbf{p}_n + (1-\lambda'_n)\mathbf{p}_2 > \mathbf{p}_1.$$

Thus, we effectively substituted the last three inequalities of the system with $n+1$ prices by the last two inequalities for the system with only $n$ prices. From the induction hypothesis, we know that this system has no feasible solution. This infeasibility finishes the sufficiency part of our proof, since we can conclude that the triangular configuration implies WARP-reducibility.

**Necessity.**To show the reverse, let us consider a set of prices $P$ that is not a triangular configuration. In particular, let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ be three distinct price vectors such that none of the vector inequalities is satisfied. First of all, as the triangular configuration is not satisfied, it must be that the three prices form the vertices of the convex set $CM(\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\})$.

Consider the convex sets $CM(\{\mathbf{p}_1, \mathbf{p}_2\})$ and $C(\{\mathbf{p}_1, \mathbf{p}_3\})$. Then,

$$CM(\{\mathbf{p}_1, \mathbf{p}_2\}) \cap (C(\{\mathbf{p}_1, \mathbf{p}_3\}) \setminus \{\mathbf{p}_1, \mathbf{p}_3\}) = \emptyset.$$

In order to see this, assume, towards a contradiction, that there exists a vector $\mathbf{p}$ such that $\mathbf{p} \geq \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$ and $\mathbf{p} = \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_3$ (with $\alpha \in ]0,1[$). Then, substitution gives

$$(\alpha - \lambda)\mathbf{p}_1 + (1-\alpha)\mathbf{p}_3 \geq (1-\lambda)\mathbf{p}_2.$$

This implies that $\lambda \neq 1$, since otherwise $\mathbf{p}_3 \geq \mathbf{p}_1$ which contradicts with $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ not being a triangular configuration. If $\alpha \geq \lambda$, then

$$\frac{(\alpha-\lambda)}{1-\lambda}\mathbf{p}_1 + \frac{(1-\alpha)}{1-\lambda}\mathbf{p}_3 \geq \mathbf{p}_2.$$

This shows that a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$ is larger than $\mathbf{p}_2$, which again implies that the prices form a triangular configuration. On the other hand, if $\lambda > \alpha$, then

$$\mathbf{p}_3 \geq \frac{\lambda-\alpha}{1-\alpha}\mathbf{p}_1 + \frac{1-\lambda}{1-\alpha}\mathbf{p}_2.$$

This shows that $\mathbf{p}_3$ is larger than the convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$, again showing that the prices form a triangular configuration. This proves our conjecture.

Therefore, from the supporting hyperplane theorem, we know that there exists a hyperplane $H(\mathbf{q}_1)$ with $\mathbf{p}_1\mathbf{q}_1 = 1$, $1 < \mathbf{p}_2\mathbf{q}_1$ and $1 \geq \mathbf{p}_3\mathbf{q}_1$. We can of course repeat this reasoning, by exchanging the indices, in order to show that there also exist a $\mathbf{q}_2$ and a $\mathbf{q}_3$ satisfying similar constraints. All this implies that there exist $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ such that

$$\begin{aligned}
&\mathbf{p}_1\mathbf{q}_1 = 1, \mathbf{p}_2\mathbf{q}_2 = 1, \mathbf{p}_3\mathbf{q}_3 = 1,\\
&1 \geq \mathbf{p}_1\mathbf{q}_2, 1 \geq \mathbf{p}_2\mathbf{q}_3, 1 \geq \mathbf{p}_3\mathbf{q}_1,\\
&1 < \mathbf{p}_1\mathbf{q}_3, 1 < \mathbf{p}_2\mathbf{q}_1, 1 < \mathbf{p}_3\mathbf{q}_2.
\end{aligned}$$

This implies a cycle of length 3 that violates SARP, while there is no WARP violation. □

## A.2 Proof of Proposition 3

*Proof.* **Necessity.** Assume that $P$ satisfies the triangular condition. We need to show the existence of two vectors $\mathbf{r}_1, \mathbf{r}_2 \in P$ such that, for all $\mathbf{p} \in P$,

$$\mathbf{p} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2,$$

where $\alpha, \beta \geq 0$ are not both zero.

Take any $\mathbf{p} \in P$. Since $\mathbf{p} \in \mathbb{R}^m_{++}$, we have that $\gamma = \frac{1}{\sum_i(\mathbf{p})_i} > 0$ and we can define $\tilde{\mathbf{p}} \equiv \gamma\mathbf{p} \in \Delta \cap P$. If $P \cap \Delta$ is a singleton, say $\mathbf{r}_1$, then we have that $\gamma\mathbf{p} = \mathbf{r}_1$, which obtains the desired result. If $P \cap \Delta$ is not a singleton, then there are at least two vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ and there exists a $j$ such that the vectors are not equal in the $j$-th component (i.e. $(\mathbf{p}_1)_j \neq (\mathbf{p}_2)_j$). Let

$$\mathbf{r}_1 \in \arg\min_{\mathbf{p}\in\Delta\cap P}(\mathbf{p})_j \text{ and } \mathbf{r}_2 \in \arg\max_{\mathbf{p}\in\Delta\cap P}(\mathbf{p})_j.$$

The compactness of $\Delta \cap P$ assures that $\mathbf{r}_1$ and $\mathbf{r}_2$ are well defined. Furthermore, by definition, we have

$$(\mathbf{r}_1)_j \leq (\tilde{\mathbf{p}})_j \leq (\mathbf{r}_2)_j \text{ and } (\mathbf{r}_1)_j < (\mathbf{r}_2)_j.$$

Since $\tilde{\mathbf{p}}, \mathbf{r}_1$ and $\mathbf{r}_2$ belong to $P$, we know that the triangular condition holds. Moreover, the inequality is actually an equality since $\tilde{\mathbf{p}}, \mathbf{r}_1$ and $\mathbf{r}_2$ belongs to the simplex $\Delta$. Indeed, suppose for instance that there exists a $\lambda \in [0, 1] : \mathbf{p} \leq \lambda\mathbf{r}_1 + (1-\lambda)\mathbf{r}_2$. If this inequality would be strict, then we obtain the following contradiction

$$1 = \sum_i (\tilde{\mathbf{p}})_i < \lambda \sum_i (\mathbf{r}_1)_i + (1-\lambda) \sum_i (\mathbf{r}_2)_i = 1.$$

A similar reasoning of course holds for the other inequalities in the triangular condition.

This shows that the triangular condition implies that there exists a $\lambda \in [0,1]$ such that one of the following three conditions hold:

$$\begin{aligned}
\tilde{\mathbf{p}} &= \lambda \mathbf{r}_1 + (1-\lambda)\mathbf{r}_2, \\
\mathbf{r}_1 &= \lambda \tilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_2, \\
\mathbf{r}_2 &= \lambda \tilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_1.
\end{aligned}$$

Note that if $\lambda = 0$ or $\lambda = 1$, these conditions imply that either $\mathbf{p} = \mathbf{r}_1$, $\mathbf{p} = \mathbf{r}_2$ or $\mathbf{r}_1 = \mathbf{r}_2$. The latter contradicts with the definition of $\mathbf{r}_1$ and $\mathbf{r}_2$, while in the first two cases we obtain what we needed to prove.

Let us then show that the last two conditions can never hold if $0 < \lambda < 1$. Assume that $\mathbf{r}_1 = \lambda \tilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_2$ holds. Then $(\mathbf{r}_1)_j \leq (\tilde{\mathbf{p}})_j$ is equivalent to $\lambda(\tilde{\mathbf{p}})_j + (1-\lambda)(\mathbf{r}_2)_j \leq (\tilde{\mathbf{p}})_j$. This implies that $(\mathbf{r}_2)_j = (\tilde{\mathbf{p}})_j$ and thus also that $(\mathbf{r}_1)_j = (\mathbf{r}_2)_j$. This contradicts with the definition of $\mathbf{r}_1$ and $\mathbf{r}_2$. A similar reasoning holds for the last condition.

As such we can conclude that $\tilde{\mathbf{p}} = \lambda \mathbf{r}_1 + (1-\lambda)\mathbf{r}_2$ and thus that $\mathbf{p} = \frac{\lambda}{\gamma}\mathbf{r}_1 + \frac{1-\lambda}{\gamma}\mathbf{r}_2$. Both coefficients are positive and at least one is different from zero.

**Sufficiency.** Take any three vector $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and assume that

$$\begin{aligned}
\mathbf{p}_1 &= \alpha_1 \mathbf{r}_1 + \beta_1 \mathbf{r}_2, \\
\mathbf{p}_2 &= \alpha_2 \mathbf{r}_1 + \beta_2 \mathbf{r}_2, \\
\mathbf{p}_3 &= \alpha_3 \mathbf{r}_1 + \beta_3 \mathbf{r}_3.
\end{aligned}$$

We need to show that the triangular condition is satisfied. Assume that $(\alpha_i, \beta_i > 0, i = 1,2,3)$. If one or more of these coefficients are zero, the reasoning is similar but the equations have to be somewhat adjusted. From the first two equations it follows that

$$\alpha_2 \mathbf{p}_1 - \alpha_1 \mathbf{p}_2 = (\beta_1 \alpha_2 - \beta_2 \alpha_1)\, \mathbf{r}_2,$$

If $\beta_1 \alpha_2 - \beta_2 \alpha_1 = 0$, then $\mathbf{p}_1$ is proportional to $\mathbf{p}_2$ and thus the triangular condition is satisfied. Else, we obtain

$$\frac{\alpha_2 \mathbf{p}_1 - \alpha_1 \mathbf{p}_2}{\beta_1 \alpha_2 - \beta_2 \alpha_1} = \mathbf{r}_2,$$

and similarly

$$\frac{\beta_1 \mathbf{p}_2 - \beta_2 \mathbf{p}_1}{\beta_1 \alpha_2 - \beta_2 \alpha_1} = \mathbf{r}_1.$$

Substituting this in the third equation then gives

$$\mathbf{p}_3 = \alpha_3 \left( \frac{\beta_1 \mathbf{p}_2 - \beta_2 \mathbf{p}_1}{\beta_1 \alpha_2 - \beta_2 \alpha_1} \right) + \beta_3 \left( \frac{\alpha_2 \mathbf{p}_1 - \alpha_1 \mathbf{p}_2}{\beta_1 \alpha_2 - \beta_2 \alpha_1} \right)$$
$$\Leftrightarrow (\beta_1 \alpha_2 - \beta_2 \alpha_1) \mathbf{p}_3 = (\alpha_3 \beta_1 - \beta_3 \alpha_1) \mathbf{p}_2 + (\beta_3 \alpha_2 - \alpha_3 \beta_2) \mathbf{p}_1.$$

We can always rearrange this last expression such that all the coefficients are positive. Therefore w.l.o.g. we can assume that there exists $\gamma_1, \gamma_2, \gamma_3 \geq 0$ (and 2 of the three distinct from zero) such that

$$\gamma_3 \mathbf{p}_3 + \gamma_2 \mathbf{p}_2 = \gamma_1 \mathbf{p}_1,$$

and also

$$\frac{\gamma_3}{\gamma_3 + \gamma_2} \mathbf{p}_3 + \frac{\gamma_2}{\gamma_3 + \gamma_2} \mathbf{p}_2 = \frac{\gamma_1}{\gamma_3 + \gamma_2} \mathbf{p}_1.$$

If $\frac{\gamma_1}{\gamma_3 + \gamma_2} \geq 1$, then $\mathbf{p}_1$ is smaller than some convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. Else $\mathbf{p}_1$ is bigger than some convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. □

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